PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
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The Radio-Micrometer

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V. *The Radio-Micrometer.*

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In the preliminary note on the Radio-micrometer which I had the honour to present to the Royal Society last year (1887), I promised to complete, as far as I might be able, the development of the instrument, and, in case of any great improvement in the proportions of the parts, to exhibit an instrument in the improved form. In the present paper I have shown how the best sizes of the several parts may be determined, and how the best result may be attained.

I must, however, first refer to the fact that on February 5, 1886, M. d'Arsonval showed, at a meeting of the Physical Society of France, an instrument called by him the Thermo-galvanometer, with which mine is in all essential respects identical. The invention of an instrument for measuring radiant heat, in which one junction of a closed thermo-electric circuit suspended in a strong magnetic field is exposed to radiation, is due entirely to M. d'Arsonval, and I need hardly say that it was in ignorance of the fact that he had preceded me that my communication was made to the Royal Society. As soon as I became acquainted with M. d'Arsonval's work, I took the earliest opportunity of admitting his claim to priority (see 'Nature,' vol. 35, p. 549).

I venture, however, to think that, although the differences between M. d'Arsonval's thermo-galvanometer and my radio-micrometer are essentially differences of detail, that even at the time of my original communication I had succeeded in producing the most sensitive instrument of practical utility, with the exception perhaps of the bolometer, which had up to that time been constructed for the measurement of radiant energy. As I hope to be able to show in the present communication that I have still further improved the proportions of the several parts, it may perhaps be fortunate that I was not aware that so able and ingenious a physicist had already made an instrument with the properties which I regarded as of so much importance, viz., the low resistance and small moment of inertia of the circuit, the small capacity for heat of the junction, the quickness and dead-beat character of the indications, and its freedom from extraneous influences. Had I known of M. d'Arsonval's work, I should probably have given no more attention to the idea.

It may, perhaps, be worth mentioning that I was led to attempt to make some improvement in the thermopile in consequence of the extraordinary sensibility which Professor LANGLEY had given to the bolometer, because it seemed incredible that, with so small a temperature coefficient of resistance as metals possess, an instrument depending on change of resistance should compare favourably with one equally well carried out, in which the comparatively large electromotive force of a thermo-electric junction is made use of.

The first object obviously was to reduce the mass of the exposed part of the pile. I was, therefore, naturally led to the use of fine wires or thin plates connected with a galvanometer in the usual way, but all the forms which such an instrument might take seemed to promise very little after the idea of the suspended circuit occurred to me. I, therefore, put them on one side, and devoted myself to the perfection of the instrument which forms the subject of the present paper. I notice, however, that there is an account of an instrument of such a kind in the January Number of the 'Philosophical Magazine' of this year (1888).

The considerations which led to the use of the general form of circuit, *i.e.*, one composed of a thin flat bar of antimony and bismuth, having its ends connected by a thin copper wire, which forms the remaining sides of a square or rectangle, are very simple.

A pair of metals must be chosen which have a high thermo-electric power, and which are not to any great extent magnetic, and which can be made exceedingly thin. Of ordinary metals antimony and bismuth so excel others in thermo-electric power that, unless they fail in other respects, they will be the best for the purpose. The diamagnetism is but a small disadvantage, and this may be overcome. The great density of the metals is objectionable; further, the difficulty of making the circuit increases as these metals are made thinner. The low conductivity for electricity of these metals is also a disadvantage, but this is balanced by their correspondingly low conductivity for heat. The disadvantages seem to be more than outweighed by the great thermo-electric power of the combination, if no attempt is made to complete the circuit with these metals only, but if copper is used for this purpose. In this way, the strongly diamagnetic metal is kept out of the intense part of the field; the high conductivity compared to its mass of copper, in which respect aluminium only is superior (but this cannot be soldered), is made use of to convey the current from one end of the bar to the other, round a circuit which may be of sufficient extent to enclose a large area in the magnetic field. Thus, the copper part of the circuit may have less weight and less resistance than it would have if made of antimony and bismuth.

As to the form of this hoop of copper, mechanical considerations determine that it shall be rectangular, for the pole pieces and central core could not conveniently be made to suit a circuit of other form; otherwise it would appear that a circular or elliptic form would be preferable.

I have attached the circuit to the lower end of a very thin capillary glass tube about 6 cm. long, which hangs at the end of a quartz fibre, made by the bow and arrow process described in the 'Philosophical Magazine' of June, 1887. Close to the

Fig. 1.

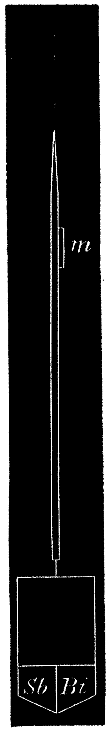


Fig. 2.

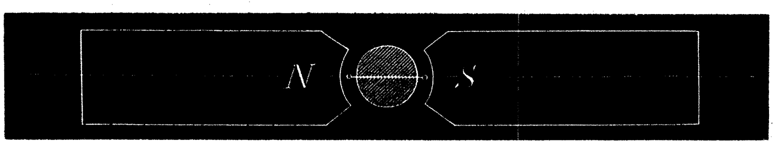
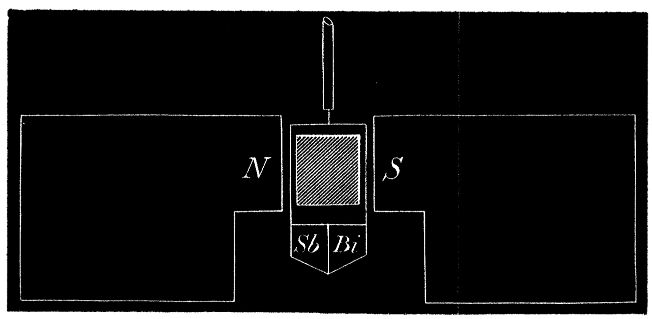


Fig. 3.



top of the glass tube is fastened a very light galvanometer mirror, m, so that the heat which may fall upon it shall have no influence on the junction lower down (see fig. 1).

The form of pole pieces which I have used is shown in figs. 2 and 3, which are a

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plan and a vertical section through the dotted line respectively. The central drum of iron, which serves to intensify and make more uniform the magnetic field, is shown shaded in these figures.

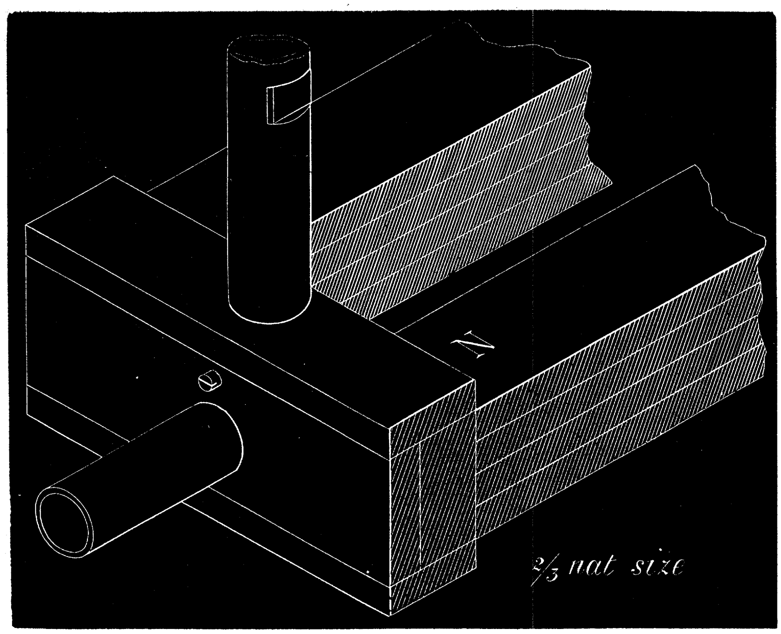
The copper hoop works in the annular space between the pole pieces and the central drum, where the field is most intense, while the active bar of antimony and bismuth hangs in the large cavity below the drum, where it is exposed to the radiation to be measured and where the weakness of the field prevents the diamagnetism from giving trouble. The pole pieces are screwed to massive brass plates, which cover the upper, front, and lower sides. The ends are left bare, as is the whole of the other side, which rests against the flat ends of a powerful horseshoe magnet. The remaining uncovered part between the poles of the magnet is covered with a piece of glass, which enables one, when levelling the instrument, to see if the circuit is free, or, when directing a spectrum upon the junction, to see that the desired colour is in its right place. It also protects the junction from air currents. Through the brass plate forming the front of the instrument, exactly opposite the junction, passes a brass tube open at the ends. This may carry a long tube, such as Professor LANGLEY used in the bolometer, fitted with gradually diminishing diaphragms, which effectually prevent air-currents from penetrating into the chamber. It should also carry, just in front of the active plate, a vertical slit, so as to confine the received radiation to the line of junction and as small a distance on either side as may be desired. A screw passing through the front plate holds the iron drum securely and truly in its place. The upper brass plate is pierced by a brass tube, about 31 cm. long, in which there is a window at the level of the mirror. There is a simple form of torsion head at the upper end of the tube which carries the quartz fibre (see fig. 4).

I should here point out in what respects my instrument differs from that of M. D'ARSONVAL. He makes his circuit of a pair of wires with the two junctions in the axis of motion, one above and one below. The wires are made of palladium and silver. The circuit is hung by a fibre of silk, and is directed by the action of the magnet on a small piece of iron wire attached to the circuit. *One of the junctions is protected by fixing over it the mirror which reflects the beam of light on to the scale.* The radiation to be measured is concentrated on to the other junction. M. D'ARSONVAL does not appear to have used for this purpose carefully formed pole pieces, but has simply hung the junction between the legs of a vertical horseshoe magnet, using a central hollow drum of iron within the circuit. He sometimes uses no central drum, but then he places the two wires very much nearer together, forming a long, narrow rectangle, which thus has a very small moment of inertia. He speaks of the great sensibility and quickness of his instrument, which also is dead beat.

In the preliminary note on the radio-micrometer ('Roy. Soc. Proc.,' vol. 42, p. 191), I pointed out in a provisional manner how the instrument may be made as perfect as possible by so choosing the length of the rectangle, the thickness of the copper wire, the number of turns or of junctions, the strength of the field, or the torsion of the fibre, that, if any of them are made more or less, the value of the instrument will be diminished. As I have now completed the calculations up to a point beyond which it would be difficult to go, and where I believe but little on which the perfection of the instrument practically depends remains to be found, I wish, without further delay, to explain the formulæ I have obtained.

Owing to the large number of variables, and the complicated manner in which they are involved, it would be difficult by any direct mathematical process to find the best value for every one at the same time; what I have done is to take the variations, one or two at a time, in such an order that those taken later shall require little or no modification of the results previously found.

Fig. 4.



It is evident that the quickness of the instrument, whatever proportions may be given to it, will increase as the sensitive plate is diminished in thickness; and later on it will be shown that, besides the quickness, the ultimate sensibility, *i.e.*, the deviation for a given rate of radiation, may also be increased as the thickness is diminished. It may, therefore, be taken as a fact that, the thinner the plate, the more sensitive will be the instrument. When I began the calculations I did not think it likely that a plate composed of metals so difficult to work could be made much less than $\frac{1}{4}$ mm. thick, and, accordingly, that has been assumed as one starting point. I have found, however, not the slightest difficulty in producing plates thinner than this, but I have in what follows taken this quantity as the thickness when wishing to find the numerical values given by the formulæ.

The next thing to determine upon is the general size of the circuit. If it is made large, the enclosed magnetic field will be increased; but, supposing the period of oscillation to remain invariable, the moment of torsion must be increased, and, further, the resistance may be increased. Both of these actions will reduce the sensibility. Let it be supposed that there are two instruments in all respects the same, except that the circuit of one is n times the length and is n times the width of that of the other, and let it be supposed that the thickness of the bar and of the wire of the circuit is the same in each case; the value of the enclosed magnetic fields will be as $n^2 : 1$; the moment of inertia, and therefore the moment of torsion, will be as $n^3 : 1$; and the resistance as $n : 1$. Therefore, $\frac{\text{included magnetic field}}{\text{torsion} \times \text{resistance}}$ will be as $1 : n^2$; that is, the angle of deflection will be n^2 times as great in the smaller as in the larger instrument. But there is a limit to the smallness, owing to two causes. The moment of inertia is made up of two parts, the circuit and the mirror, on which account, when the moment of inertia of the mirror becomes comparable with that of the circuit, the smaller instrument will have a greater moment of inertia, and, therefore, its sensibility on this account will be less than that given by the above rule.

The second reason why there is a superior limit to the sensibility as the instrument is reduced in size is due to the fact that the sensitive plate conducts more heat from the hot to the cold junction in the case of the shorter plate, and thus, for a given rate of radiation, the junction will not be so hot; and, hence, the current will be less than it would be if the diminished resistance were the only cause of change. It is, therefore, necessary to choose some length of plate or width of rectangle which can conveniently be made in practice, and, assuming this as a constant, to find by calculation the best values of all the variables, and so, as it were, to fit to the plate the best possible instrument. Something must be assumed as a starting point, and this seemed, on the whole, the most convenient. I have assumed the plate, then, to be composed of two squares of antimony and bismuth, each 5 mm. in the side and $\frac{1}{4}$ mm. thick, soldered edge to edge, thus forming a plate $10 \times 5 \times \frac{1}{4}$ mm. This assumption I have made simply for the sake of numerical calculation; any size may be equally well taken as the basis of operations, provided that in the equations which follow the proper numerical values are assigned to the constants.

Of the three dimensions of the plate, the length, the thickness, and the breadth, the first two only need be considered as assumptions which can in any way affect the result, for it matters not how the breadth be varied, provided that the sectional area of the wire and the moment of torsion are varied in the same proportion, *i.e.*, if the whole breadth is exposed to the radiant energy.

It will first be convenient to see how the circuit can be formed, so as to give the best results when the magnetic field is supposed constant; it will be seen later that this best is not the ultimate best when the field and the circuit are adapted to one another.

The circuit may be considered best from more than one point of view. It may be with respect to weight, or with respect to moment of inertia.

The best circuit with respect to weight would be that which would give the greatest deflection when supported by a particular bifilar arrangement free from torsion. The strength of the field, the thermo-electric power of the junction, and the value of the radiation, being constant factors, may be omitted for the present.

Let the following be the meanings of the several symbols that will be used :—

(·1481)	W.	The weight of the plate, mirror, and stem, the invariable or dead weight.
	w.	The weight of the hoop of copper, the variable weight.
($6·742 \times 10^6$)	C.	The resistance of the plate, the invariable or dead resistance.
	r.	The resistance of the hoop of copper, the variable resistance.
	l.	The length of the rectangle.
	n.	The number of turns of wire.
(·00895)	u'.	The weight of a unit piece of copper ($1 \times ·1 \times ·01$ cm.).
($1·642 \times 10^6$)	v.	The resistance of a unit piece of copper ($1 \times ·1 \times ·01$ cm.).
	a.	The sectional area of the wire ($·1 \times ·01$ being considered unit area).

The reason for taking 1/1000th of a square centimetre as the unit of sectional area for the wire is that this is not very different from the actual sizes that will be required, and that it avoids the absurdity of supposing a hoop of wire of such a size made of wire of 1 sq. cm. in sectional area. It is a matter of convenience, and nothing more. The mirror that I have used is 6 mm. in diameter, and weighs ·04 grm. The glass stem may be taken as ·005 grm. The weights and resistance of the three metals are taken from LUPTON'S Tables, and the numerical values of the several quantities so calculated are enclosed in parentheses before their respective symbols. All the quantities, except when otherwise specified, are in C.G.S. units.

It is necessary to bear in mind that a certain excess of wire may be required, over and above that actually necessary to reach the active bar. Though in the numerical examples which follow I have not allowed for any, I have taken care in the equations to introduce a symbol, p, which must be explained. Imagine a circuit made of one turn 1 cm. wide and l cm. long: then, if there is no excess, the amount of wire will be $2l + 1$; but, if there is an excess, then the amount will be $2l + 1 +$ excess. I have used the symbol p in the following calculations for $1 +$ excess, and I have assumed that both the resistance and the weight vary with the total length. The only possible discrepancy can be due to the manner in which the current leaves the wire for the plate.

Considering, first, the case of a circuit of only one turn of wire, the variable resistance r and weight w of the wire will be

$$r = (2l + p)\, v \div a,$$
$$w = (2l + p)\, u' \times a.$$

The conductivity G of the whole circuit will be

$$\mathrm{G} = \frac{a}{(2l + p)\, v + a\mathrm{C}}.$$

The efficacy of the circuit with respect to its weight, E_{wt}, *i.e.*, the moment which it can exert upon a unit field for every gramme that it weighs when unit E.M.F. is acting at the junction, is

$$\mathrm{E}_{wt} = \frac{l \times \mathrm{G}}{\mathrm{W} + w} = \frac{la}{\{(2l + p)\, v + a\mathrm{C}\}\{\mathrm{W} + (2l + p)\, au'\}}.$$

Now, it is evident that there must be a maximum value for E_{wt}, both when l and when a is varied, for, if either is made very great or very small, more is lost than is gained. If, therefore, the expression for E_{wt} is treated in the usual way to find the two maxima, the result will be found to be

$$\text{best } a = \sqrt{\frac{\mathrm{W}v}{u'\mathrm{C}}} \quad \ldots\ldots\ldots\ldots \quad (1),$$

$$l^2 = \frac{(\mathrm{W} + pau')(pv + a\mathrm{C})}{4vau'};$$

or, substituting the value of a above, it will be found that

$$2l = p + \sqrt{\frac{\mathrm{CW}}{u'v}} \quad \ldots\ldots\ldots\ldots \quad (2).$$

If, further, the circuit is supposed to have n turns, the corresponding expressions will be

$$a = \sqrt{\frac{\mathrm{W}v}{u'\mathrm{C}}},$$

$$2nl = (2n - 1)\, p + \sqrt{\frac{\mathrm{CW}}{u'v}} \quad \ldots\ldots\ldots \quad (3).$$

Thus, the size of wire that is most suitable is independent of the length of the rectangle or of the number of turns. The numerical values of these quantities for 1 . . . 5 turns are as follows:—

n.	Best l.	E_{wt}.	Best a.
1	4·621	$9{\cdot}202 \times 10^{-7}$	2·0075.
2	2·811	7·565 ,,	
3	2·207	6·423 ,,	
4	1·905	5·580 ,,	
5	1·724	4·933 ,,	

Thus, it appears that one turn is better than any other number, provided that the circuit may have sufficient length, and, of course, that the magnetic field is sufficiently extended. This result is really self-evident, for, whatever may be the efficacy of a circuit of say two turns of the best length, one of one turn of twice the length must be better, as in this case the value of the enclosed magnetic field will be the same, while the resistance and the weight will each be less, and, further, the circuit of twice the length will not have the best length for one of one turn only.

If the length of the circuit is limited to 1 cm., then two turns are better than one or three. The series of figures found on this supposition are not of sufficient importance to be worth giving.

It is interesting, however, to notice how slowly the efficacy changes when the length of the circuit is not the best. The following figures show this :—

l.	E_{wt}.	
3	$8{\cdot}742 \times 10^{-7}$	
3·5	9·036 ,,	
4	9·152 ,,	
4·5	9·200 ,,	
4·621	9·202 ,,	max.
5	9·190 ,,	

Thus, it is not a matter of much consequence whether the circuit is very near the best length or not.

If it is desired to find the numerical value of the resistance or the weight of the wire part of the circuit when it is of the best length and sectional area, the following expressions, which have been obtained by substituting in those for r and w the best values found for a and l, may be used to save time :—

$$\text{best } r = C\left(1 + 2p\sqrt{\frac{u'v}{CW}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

$$\text{best } w = W\left(1 + 2p\sqrt{\frac{u'v}{CW}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

From these it is seen that that length and size of wire is best of which the weight as much exceeds the dead weight as the resistance exceeds the dead resistance. In the same way, the efficacy of the best circuit may be shown to be

$$\text{best } E_{wt} = \frac{1}{8\sqrt{(\mathrm{CW}u'v)} + pu'v} \quad \ldots \ldots \ldots (6).$$

If, finally, the breadth be made a variable, the following equations will give the best conditions :—

As before,

$$\text{best } a = \sqrt{\frac{wv}{u'\mathrm{C}}};$$

and this is true whatever length, number of turns, width, or even shape the circuit may have. When b is greater than 1,

$$\text{for the best length, } 2l = 2b - 1 + e + \sqrt{\frac{\mathrm{CW}}{u'v}},$$

$$\text{for the best breadth, } 2b = 2l - 1 + e + \sqrt{\frac{\mathrm{CW}}{u'v}},$$

where e is the excess used for soldering.

Therefore, for any breadth the length must exceed the breadth by as much as the breadth should exceed the length when that is given. In other words, the circuit is improved by adding to it *ad infinitum*, so that a square of infinite size is the best rectangle. If it should happen that $\sqrt{(\mathrm{CW}/u'v)} + e < 1$, then the circuit in the same way would be made worse by increasing the dimensions. As a matter of fact, in the particular case, taking e as 0, this quantity is equal to 8·24.

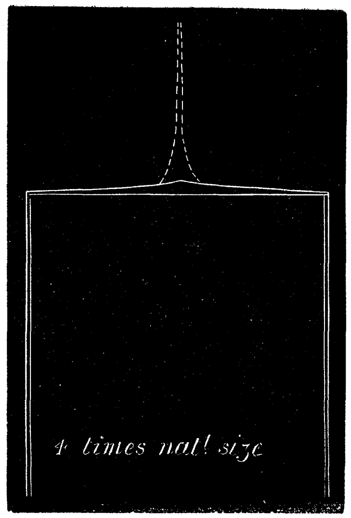
If, on the other hand, b is less than 1, then the quantity $2b - 1 + e$ in the two equations above must be replaced by $b + 1 + e$, a quantity necessarily positive; hence, whether b is less than or greater than 1, the circuit cannot be too large.

If the same process that has been followed in finding the best conditions with respect to weight be employed to find them with respect to moment of inertia, a difficulty arises in consequence of the fact that the upper end or cross wire of the circuit has a resistance which depends upon its length simply, while it has a moment of inertia which is only one-third of what it would have if it were placed alongside of one of the side wires of the circuit. Thus, while the expression for r remains as before, viz., $(2l + 1 + e)\, v \div a$, that for the moment of inertia of the wire will be $\{(6l + 3e + 1)/12\}\, u'a$. If with these values the attempt is made to find the best values for a and l with respect to moment of inertia, a complicated cubic equation results and symmetrical expressions can no more be obtained. If, however, the two coefficients in parentheses had the same value, there would be no difficulty.

While trying to find a remedy for this difficulty, I noticed that a wire of uniform section is not the best form of conductor when the moment of inertia is taken into account, *i.e.*, of that part of the wire which crosses the axis. That this is so is evident in two ways. Since a wire of uniform section has the least resistance for a given length and weight, it cannot have the least resistance for a given length and moment of inertia, except when it is parallel to the axis of rotation. Or, considering the cross piece only, since the uniform section is that which gives the least resistance for its weight, it is clear that a very small change of form, such for instance as would occur if a thin skin were removed from the outer and were transferred to those portions nearer the axis of rotation, would produce no appreciable change in resistance, but it would reduce the moment of inertia; therefore, there is an advantage in having those parts near the axis thicker than those more distant from it. If we suppose that the resistance of any element of the wire between the axis and the ends is inversely proportional to the cross section, and that the resistance of the whole piece is the sum of the resistances of the several elements, which is true as long as the rate of change of section is not so rapid as to make the stream lines notably inclined, then the best variation of section will be that in which no change will be made in the resistance ÷ the moment of inertia if a thin skin be transferred from one part to another. Taking the axis as origin of rectangular coordinate, and calling the sectional area y, then for all values of x, $1/y \div yx^2$, *i.e.*, $1/y^2x^2$ must be constant or y must vary inversely as x. It thus appears that, if the hoop is cut out of sheet copper instead of wire, that part which forms the upper bar should be bounded by two hyperbolas of such dimensions that the width at the ends of the cross bar is the same as that of the side pieces, which are uniform. If the metal is chosen of such a thickness, t, that the best section of the sides hereafter to be found $= t \times \cdot 1$ mm., then the error due to the inclination of the stream lines already pointed out will not be appreciable, except within about $\frac{1}{2}$ mm. on either side of the axis. It will be well at once to point out that the value of this supposed form for the cross bar does not at all lie in the fact that it is the best form, for nothing worth consideration would be gained by adopting it, but the reason for bringing it forward is this: the total resistance of either half of the cross bar will be, if the resistance of the same length of the wire be called $1, = \int_0^1 x\,dx = \frac{1}{2}$; the moment of inertia of either half will be, if that of the same length of the side be called $1, = \int_0^1 x\,dx = \frac{1}{2}$. That is, not only has the best form been found for the cross bar, but the coefficients for moment of inertia and for resistance have at the same time been made identical, and thus all the expressions found with respect to weight will equally apply with respect to moment of inertia.

It may be objected, since, as already mentioned, the resistance of these parts close to the axis is greater than is supposed, on account of the inclination of the stream lines and the consequent concentration of the current in the more direct line, or again, since the weight of a piece of metal filling the space between two hyperbolas is infinite, that the solution for the difficulty thus put forward is not correct, and that any further calculations based upon this result will not be trustworthy. The answer evidently is that, if the axial spike be removed altogether, as shown in fig. 5, the actual resistance and moment of inertia would not be so much as 1 per cent. different from the value ($\frac{1}{2}$) already found, and that, as these calculations are merely a guide to direct the design of a practical instrument, an assumption which is not more than 1 or 2 per cent. in error in what is after all a small fraction of the total resistance and moment of inertia—one which, in fact, is so true that neither the power of the instrument maker nor our knowledge of the constants will prevent our introducing in practice variations of far greater magnitude—may be allowed to pass, especially as without something equivalent it would be impossible to find expressions which would apply to the more complicated conditions which will be considered later.

Fig. 5.



The same considerations apply exactly to the sensitive plate, which, when the greatest efficacy with respect to moment of inertia is required, should no longer be made prismatic, but should become more narrow towards the ends, as shown in fig. 1. I may mention that in the original instrument, which I made before I had arrived at any of these results, I did, as a matter of fact, make the active bar of the shape shown, because I felt that the lower corners were doing more harm by their moment of inertia than they were doing good by their conductivity.

Since the active plate is taken as one of the invariables of the circuit, any proportions that may be desired may be taken for it, and the resulting resistance and moment of inertia made use of in calculating numerical values. I have in the arithmetical work which follows taken the active plate to be of the original shape and size, that is, a rectangular prism, $10 \times 5 \times \cdot 25$ mm. I have further not made any allowance for excess of metal for soldering, *i.e.*, p has been taken as the coefficient of resistance and moment of inertia of the upper bar only, which has been found to be equal to $\frac{1}{2}$, just as, when the conditions with respect to weight were considered, p was taken as equal to 1.

As in the rest of this paper the best conditions with respect to weight will no more be considered, but only those relating to moment of inertia, it will be well here to specify the meaning of the several symbols that will be used. The arithmetical values of the fixed quantities are, as before, included within parentheses.

$(9{\cdot}49 \times 10^{-3})$	K.	The moment of inertia of plate, mirror, and stem, the dead moment of inertia.
	k.	The moment of inertia of the copper hoop, the variable moment of inertia.
$(6{\cdot}742 \times 10^{6})$	C.	The resistance of the plate, the dead resistance.
	r.	The resistance of the copper hoop, the variable resistance.
	l.	The length of the rectangle.
	n.	The number of turns of wire.
$(2{\cdot}2375 \times 10^{-3})$	u.	The moment of inertia of the unit piece of copper $(1 \times {\cdot}1 \times {\cdot}01)$ at 5 mm. from the axis.
$(1{\cdot}642 \times 10^{6})$	v.	The resistance of the unit piece of copper.
	a.	The sectional area of the wire (${\cdot}1 \times {\cdot}01$ being considered unity).

Since the formulæ already given are now equally true with respect to moment of inertia, it will be unnecessary to do more than barely state them here for future reference.

$$\text{For best sectional area } (n \text{ turns}) \; a = \sqrt{\frac{\mathrm{K}v}{u\mathrm{C}}} \quad \ldots \ldots \quad (7),$$

$$\text{for best length } (n \text{ turns}) \; 2nl = (2n - 1)\,p + \sqrt{\frac{\mathrm{KC}}{uv}} \quad \ldots \ldots \quad (8),$$

$$\text{for best } r \text{ (one turn) } r = \mathrm{C}\left(1 + 2p\sqrt{\frac{uv}{\mathrm{CK}}}\right) \quad \ldots \ldots \quad (9),$$

$$\text{for best } k \text{ (one turn) } k = \mathrm{K}\left(1 + 2p\sqrt{\frac{uv}{\mathrm{CK}}}\right) \quad \ldots \ldots \quad (10),$$

$$\text{for best } \mathrm{E}_k \text{ (one turn) } \mathrm{E}_k = \frac{1}{8\,\{\sqrt{(\mathrm{CK}\,uv)} + puv\}} \quad \ldots \ldots \quad (11).$$

$p = \frac{1}{2} +$ any excess allowed for soldering.

The following arithmetical results are obtained from some of these expressions, making $p = \frac{1}{2}$:—

n.	Best l.	E_k.	Best a.
1	2·3366	$7·1863 \times 10^{-6}$	1·0174.
2	1·4183	3·4222 ,,	
3	1·1122	2·1208 ,,	
4	·9591	1·5014 ,,	
5	·8673	1·1500 ,,	

Thus, the best length and the best sectional area with respect to moment of inertia are each just over half the values found with respect to weight. One turn, of course, gives a greater efficacy than any other number, just as it did before, and for the same reason, but in this case one turn is more than twice as good as two, whereas it was only about one-fifth in excess.

The result at present obtained, then, is this: provided a torsion fibre of convenient length can be obtained which shall in every case produce a certain pre-determined period, say 10 seconds, which, with my process for quartz, is possible even when the hanging body is far lighter and smaller than any galvanometer mirror, then the moment of torsion and the moment of inertia will always have the same ratio; therefore, the circuit which produces the greatest couple for its moment of inertia will also produce the greatest couple compared with the torsion, and will, therefore, give the greatest deflection. This circuit has been found above to have a length of 2·3366 cm., and to be made of copper, having a sectional area of ·0010174 sq. cm. at the sides.

Though this circuit has the best length and the best sectional area under certain magnetic conditions, it is not necessarily the best when these are varied.

It has, of course, been understood that in the comparison between long and short circuits the magnetic field is in both cases the same, and is uniform throughout the whole of the area enclosed by the circuit. This may or may not be true when an internal pole piece is arranged as shown in figs. 2 and 3. There must be some clearance above and below; thus, the internal drum is always less than the actual length. It is therefore probable that the effective area is proportional to the length, less a small constant quantity. If this defect is equal to the sum of the clearances above and below the drum, then the length of the drum must be called l in the preceding formulæ, and four times the clearance added to p. It is not possible to say exactly what is the error thus introduced into the calculations, but in no case can it be important. This, however, is a mere detail compared with the effect of very strong magnetism.

When a closed circuit oscillates under the conditions in which that in the instrument is placed, induced currents are formed which oppose the motion, and thus, if the field is strong enough, or the conductivity great enough, the oscillatory character ceases, and the circuit slowly moves towards its resting place, more slowly as it approaches it. It is a great advantage in an instrument that it should in this way be dead beat; but, if the field is more than strong enough, the extra resistance to the motion so increases the time of coming to the resting place that the loss of time more than counterbalances any advantage given by the increased ultimate sensibility. If the ultimate sensibility is required to be made a maximum, then the expressions found are the best, and, the stronger the field, the better; but, if the best combination which is dead beat (and no more than dead beat) is required—and this is what any one who has used both would require—then the length and the sectional area already found are not the best, unless the strongest field which can conveniently be employed is not strong enough to make the motion dead beat. In the particular case it is more than strong enough.

It is necessary, therefore, to introduce the effect of another variable, the strength of the field, the relation between it and the rest of the circuit being such that the motion is just dead beat.

It is well known that motion ceases to be oscillatory when half the coefficient of resistance to the motion is equal to the square root of the acceleration when the angular displacement is unity. In the investigations depending on these relations the following symbols will have the meanings attached to them :—

S. Sensibility, *i.e.*, efficacy $\times$ enclosed magnetic field.

A. Area enclosed by the circuit.

ω. Angular velocity.

T. Time of a complete undamped oscillation.

G. Conductivity of the whole circuit.

α. Angle included between the plane of the circuit and the direction of the lines of force, supposed parallel to one another.

κ'. Moment of inertia of the whole circuit, *i.e.*, $K + k$.

H′. The minimum strength of magnetic field for which the motion is dead beat. This will hereafter be called the dead beat magnetic field.

The resistance to the motion of the circuit

$$= G \times H^2 \times A^2 \times \omega \times \cos^2 \alpha ;$$

half the coefficient of resistance $= GH^2A^2 \cos^2 \alpha / 2\kappa'$.

Since the magnetic field is radial and is everywhere cut normally by the side wires of the circuit, the factor $\cos^2 \alpha$ ought to be omitted. Owing to the small possible angular deflection, it could not in any case differ appreciably from 1.

$$\sqrt{(\text{acceleration at unit angle})} = \frac{2\pi}{T}.$$

Since, when the motion is just dead beat, half the coefficient of resistance is equal to $\sqrt{(\text{acceleration at unit angle})}$,

$$\frac{GH'^2A^2}{2\kappa'} = \frac{2\pi}{T},$$

or

$$H' = \frac{2}{A}\sqrt{\frac{\pi}{T}} \cdot \sqrt{\frac{\kappa'}{G}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12),$$

$$S = \frac{GAH'}{\kappa'} = 2\sqrt{\frac{\pi}{T}} \cdot \sqrt{\frac{G}{\kappa'}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

On differentiating the expression for S with respect to a (the sectional area of the wire) and equating to 0, the maximum sensibility will be found when a has the same value $\sqrt{(Kv/uC)}$, which gave the maximum efficacy; on the other hand, the differential coefficient with respect to l (the length of the rectangle of copper) is negative for all positive values of l, that is, however short l may be, provided that the magnetic field may be made strong enough to keep the motion dead beat, a still greater sensibility will be given by a shorter circuit. If the strongest magnetic field conveniently available is not sufficient to make the circuit having maximum efficacy dead beat, then that circuit is still the best. If, however, as will be the case with an ordinary magnet, the field is more than strong enough, then the length of rectangle must be reduced until the motion is deat beat. Taking the particular arrangement already referred to, in which the circuit is 1 cm. wide and is made of wire having the best sectional area (·001017 sq. cm.), the following are the values of the dead beat magnetic field for various lengths of circuit :—

								Best.
Length . .	·2	·4	·6	·8	1·0	1·5	2·0	2·3366
H′	1722	929	664	532	455	347	294	271·8

It is interesting here, in the expression for H′ (12), to substitute those values of κ' and G which belong to the circuit of greatest efficacy. If this is done, there finally results the equation

$$H'' = 8\sqrt{\frac{\pi}{T}}\sqrt{uv} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12a),$$

where H″ is the dead beat magnetic field for the circuit of greatest efficacy. It thus appears that, no matter what the resistance of the antimony-bismuth bars, or what the moment of inertia of these bars, the mirror, and stem may be, provided that the circuit is so formed as to produce the greatest ultimate sensibility in any given field, the motion will be dead beat for one particular strength which simply depends upon the specific gravity and the specific resistance of the material with which the circuit is completed.

The sensibility obtained by such a combination is

$$S = \sqrt{\frac{\pi}{T}} \cdot \frac{1}{\sqrt{CK} + p\sqrt{uv}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13a).$$

As has already been shown, this is not the best arrangement to make use of. That is best in which the circuit is the shortest which will remain dead beat in the strongest magnetic field available.

In the original instrument, of which figs. 2 and 3 show the pole pieces, a strong compound horseshoe magnet was used. The working field was tested by replacing the active circuit by one composed of 50 turns of 1 sq. cm. each of the finest insulated copper wire. This was mounted so that it could be suddenly twisted through a definite angle by moving an arm between a pair of stops. The ends of the coil were connected with a ballistic galvanometer in a distant room, and the throw observed. The resistance of the whole circuit was measured. Then, in the place of the coil, a condenser of known capacity and a cell of known E.M.F. were arranged with a key so that a definite discharge of electricity could be sent through the galvanometer. The absolute value of that sent by each oscillation of the coil could thus be determined, and hence the value of the field found. In this way it was found that without the keeper a field of 1342 units existed in the working space.

It appears, then, from the table on the previous page, that with such a field the circuit should be about 3 mm. long. I did not actually alter the shape of the circuit, but adjusted the field by means of a sliding armature, until the sensibility, or the resistance to the motion, produced a convenient result. As has been shown, this is not so good a plan as adapting the circuit to the strongest field that is available, though less is lost by reducing the field than might be expected, as will be explained later.

If the breadth b that will give the greatest efficacy with respect to moment of inertia is required, there is no difficulty in finding the best a and l in terms of b, but the best b is involved in an expression of such complexity that it can only be found by arithmetical means.

It is not worth while to give at length the table showing the successive values of the efficacy, as b varies; it is sufficient to state that not only is the efficacy diminished by increasing the breadth beyond that of the active bar, but it is even increased as the breadth diminishes down to 2 mm., and probably far beyond.

The conclusion, then, is obvious, that not only should the rectangle be as narrow as possible, but the junction should be arranged also in a correspondingly narrow form. Further, in consequence of the extreme narrowness of the circuit, the resistance and moment of inertia of the cross wire may practically be neglected in comparison with the now much greater length of the rectangle. I must here remark that I have thus been brought to the adoption of the excessively narrow form which M. D'ARSONVAL has used. I do not know whether he was aware that the narrow form is not only far quicker than a wide form, which is the reason he gives for adopting it, but that it is in addition, when a convenient period is arranged, also more sensitive. I certainly did not expect to find it so.

Making use of the same methods and symbols, but neglecting the now infinitesimal effect of the cross wire, the following equations will be found to hold :—

$$\text{best} \quad a = \frac{1}{b}\sqrt{\frac{\mathrm{K}v}{u\mathrm{C}}} \quad \ldots \ldots \quad (14),$$

$$\text{best} \quad l = \frac{1}{2b}\sqrt{\frac{\mathrm{KC}}{uv}} \quad \ldots \ldots \quad (15),$$

$$\text{greatest} \quad \mathrm{E}_k = \frac{1}{8\sqrt{(\mathrm{KC}uv)}} \quad \ldots \ldots \quad (16).$$

Further, it will be found that

$$\mathrm{C} = \frac{2lv}{a} \quad \text{and} \quad \mathrm{K} = 2\,laub^2,$$

and thus the copper hoop must be so proportioned that its resistance may be equal to the dead resistance, and its moment of inertia to the dead moment of inertia. It is also found that under the supposed circumstances, namely, that the cross wire is of no account, the efficacy is independent of the breadth, and the only effect of an increased breadth is to require a diminished sectional area of wire and a diminished length; thus, during variation of the breadth neither the resistance nor the moment of inertia of the copper wire is changed.

The expression for the dead beat magnetic field, when the circuit of greatest efficacy is used, is, as before,

$$\mathrm{H}'' = 8\sqrt{\frac{\pi}{\mathrm{T}}}\sqrt{uv} \quad \ldots \ldots \quad (17);$$

and this seems to be very generally true.

The sensibility under these conditions is

$$\mathrm{S} = \sqrt{\frac{\pi}{\mathrm{T}}} \cdot \frac{1}{\sqrt{\mathrm{KC}}} \quad \ldots \ldots \quad (18),$$

which shows that, no matter what the material of the hoop may be, if the dimensions which give the greatest sensibility in any given magnetic field are employed, and if the field is so strong that the motion is dead beat, the sensibility will always be the same, and this will only depend on the resistance and moment of inertia of the invariable part of the circuit.

Since, as in the case of the wide circuit, the magnetic field that will make the motion of the best narrow circuit dead beat is far less than that which is available, it will, as before, be best to employ a circuit so much smaller than the best as will just make the motion dead beat.

The circuit may be made smaller by reducing either factor, the length l or the breadth b. As has been shown, it is indifferent whether one or other factor is altered, provided that b is small compared with l. Let the product be made N times as small, so that bl becomes bl/N, then the first factor $2/bl$ in the expression for H′, which it is convenient to repeat here in another form,

$$\left(\mathrm{H}' = \frac{2}{bl}\sqrt{\frac{\pi}{\mathrm{T}}}\sqrt{\kappa'\mathrm{R}}\right),$$

will become $2\mathrm{N}/bl$. But when bl is changed, both κ', the total moment of inertia, and R, the total resistance, are changed also. In the circuit of greatest efficacy the resistance and moment of inertia of the hoop are each equal to the fixed or invariable resistance and moment of inertia; and thus, when bl becomes bl/N, one half of each κ' and R also becomes one-N^{th} of what it was, the other half of each remaining unchanged, and thus the new expression for H′ becomes $\frac{\mathrm{N}+1}{bl}\sqrt{\frac{\pi}{\mathrm{T}}}\sqrt{\kappa'\mathrm{R}}$, that is $\frac{1}{2}(\mathrm{N}+1)$ times what it was. Therefore, if it is desired to make the dead beat magnetic field M times that which has been found for the circuit of greatest efficacy, the product bl must be reduced until it is $2\mathrm{M}-1$ times as small, for $\mathrm{N} = 2\mathrm{M} - 1$.

By this process the actual sensibility is increased, and the amount of increase may be found as follows:—The sensibility of any combination varies directly as the magnetic field and as the product bl, and inversely as the total moment of inertia and the total resistance; of these four quantities it has just been shown, that when bl is divided by N, the dead beat magnetic field is multiplied by $\frac{1}{2}(\mathrm{N}+1)$, and at the same time R and κ' are each multiplied by $(\mathrm{N}+1)/2\mathrm{N}$; therefore, the sensibility of the arrangement becomes

$$\frac{\mathrm{N}+1}{2} \times \frac{1}{\mathrm{N}} \times \frac{2\mathrm{N}}{\mathrm{N}+1} \times \frac{2\mathrm{N}}{\mathrm{N}+1}, \quad \text{that is,} \quad \frac{2\mathrm{N}}{\mathrm{N}+1},$$

or $2 - (1/\mathrm{M})$ times what it was, so that

$$\mathrm{S} \text{ becomes } \left(2 - \frac{1}{\mathrm{M}}\right)\sqrt{\frac{\pi}{\mathrm{T}}}\frac{1}{\sqrt{\mathrm{KC}}}.$$

Since the dead beat magnetic field for the circuit of greatest efficacy is about 272 units, N must be so chosen as to make H′ four or five times as great. Assuming that H′ is to be increased to four times its original value, or that $\mathrm{M} = 4$, the sensibility will only become $1\frac{3}{4}$ times what it was. Even in the case of an infinite field, it cannot be more than double that due to a field of 272 units if the motion is only just dead beat. From this it appears that, as long as the circuit has dimensions which at all approximate to those which theoretically are best, the sensibility obtained by moving the pole pieces until the dead beat conditions or the desired logarithmic decrement are produced is practically the highest which is possible.

Owing to the fact that, with increase in the breadth of the circuit, the cross piece becomes increasingly mischievous, both on account of its moment of inertia and of its resistance, it is clear that the circuit cannot be too narrow until the increased length becomes such that it is inconvenient to provide a magnet and pole pieces which will enclose so great a length. In another way the thick wire which the narrow circuit requires is advantageous, as will appear shortly.

Having thus found the best relations between the variable copper and the arbitrary junction and mirror, it remains to see how these may be modified with advantage.

As, with the narrow form of circuit, the smallest galvanometer mirror has a moment of inertia many times as great as that of the active bars, and since the copper must have a moment of inertia equal to their sum, it is evident that it will be advantageous to reduce the dimensions of the mirror until it again becomes small in comparison. By this reduction the defining power of the mirror, supposed optically perfect, is also reduced, and thus there must be a limit at which as much is lost by the increasing want of definition as is gained by the diminishing moment of inertia.

The defining power of a perfect mirror—and the smaller the mirror the more likely it is to be perfect—varies with its diameter, while the moment of inertia is proportional to the fourth power of the diameter when the thickness is constant, or to the fifth power if the thickness is also proportional to the diameter. To find the best diameter it is necessary to remember that the fixed moment of inertia K is the sum of the moment of inertia of the junction K_j and of the mirror K_m. Thus, the accuracy of observing a deflection $= K_m^{1/n}/2\,(K_m + K_j)$, where $n = 4$ or 5 as the case may be. The best size of mirror then will be such that

$$K_m = \frac{K_j}{3}; \qquad k = \frac{4K_j}{3} = 4K_m, \text{ when } n = 4,$$

or that

$$K_m = \frac{K_j}{4}; \qquad k = \frac{5K_j}{4} = 5K_m, \text{ when } n = 5.$$

Using the thinnest microscope cover glass, about ·1 mm. thick, it will be found that the size of mirror which gives the best result when the antimony-bismuth bars have the dimensions which will be assigned to them hereafter is one having a diameter of $2\frac{3}{4}$ mm.

I have picked out a number of discs sufficiently thin, silvered them, cut pieces of the proper size, and then examined them by reflection. With mirrors as small as this, the eye itself takes the place of the usual telescope, and it is easy to choose those which allow the eye to see by reflection fine distant lines as clearly as if the light came direct. The only difficulty I have had in attaching these mirrors to the stem arises from the bending of the glass under the action of even the smallest quantity of cement. If sealing-wax is used—and this is the least magnetic of all the cements I have examined—a speck less than 1 mm. in diameter will, by its capillarity when melted, so distort the glass, even though it only touches near one edge, as to make a double image clearly visible. All difficulty is overcome by using instead the smallest visible quantity of shellac varnish, and applying heat to a certain extent. Of course the light reflected from so small a mirror is not sufficient when the usual paraffin lamp is employed, but, with oxygen at its present price, there is no reason why a small lime light should not be used. I have found that with even a small supply of oxygen the light on the scale is abundant, and there is no difficulty in observing a deviation of $\frac{1}{4}$ mm. The theoretical defining power upon a scale a metre distant of a mirror $2\frac{3}{4}$ mm. in diameter is about ·23 mm.

The image of the cross wire given by a mirror of this size that was used in the radio-micrometer shown to the Royal Society was a sharp line which could be read with an accuracy of $\frac{1}{10}$ mm.

As the little mirror is plane, I have cemented a plano-convex lens of a convenient focus in the place of the usual plane glass window which must be used to protect the moving parts from currents of air. This is preferable to a double convex lens, because the flat surface is more convenient for cementing, but especially because this surface by reflection also throws an image on the scale which is invariable in position, and which may be used as a reference mark if the scale is moved.

As the definition of the mirror is still so good that the power of reading a deflection in the ordinary way is not materially reduced, no change will be practically necessary in the series of equations 14–18, which are only strictly applicable when the defining power is not affected by change of K.

The junction is the only part of the suspended portion of the instrument which now remains arbitrary. I have provisionally assumed, for the sake of arithmetical results, a pair of bars of antimony and bismuth $5 \times 1 \times \frac{1}{4}$ mm. fixed parallel to one another at a mean distance apart of 1 mm. A less mean distance is impracticable, though it would be an advantage; but the length and sectional area may be modified if found necessary.

Before considering the effect of varying the proportion of the antimony-bismuth bars, it will be convenient at this point to find numerically the value of the combination (see fig. 6) which has thus been developed.

They are as follows :—

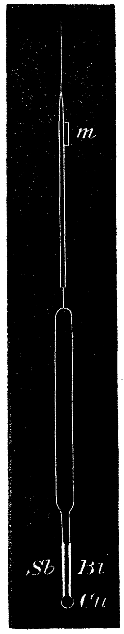
$$\left.\begin{aligned} a &= \frac{1}{b}\sqrt{\frac{\mathrm{K}v}{u\mathrm{C}}} = \cdot 000387 \text{ sq. cm.} \\ l &= \frac{1}{2b}\sqrt{\frac{\mathrm{KC}}{uv}} = 3\cdot 970 \text{ cm.} \end{aligned}\right\} \text{if } b = 1 \text{ mm.},$$

$$\mathrm{E}_k = \tfrac{1}{8}\frac{1}{\sqrt{\mathrm{KC}uv}} = 4\cdot 284 \times 10^{-5},$$

$$\mathrm{H}'' = 8\sqrt{\frac{\pi}{\mathrm{T}}}\sqrt{uv} = 271\cdot 8.$$

Now, assuming that the working field is four times that found, then the product bl must be made one-seventh of that stated above, and the product $E_k \times H''$, which

Fig. 6.



represents the available sensibility, must be made $1\frac{3}{4}$ times as great. It is satisfactory to find that the greatest efficacy, and therefore sensibility, is with the narrow circuit about six times as great as that found for the wide circuit first considered.

There is no reason why the exact dimensions assigned to the active bars should be employed; it will be well, therefore, to consider what will be the effect of using bars of other dimensions.

Let the sectional area be supposed increased in the ratio $1 : n$; then K will become nK, and C will become C/n; therefore, the greatest efficacy which depends on the product of these will be unchanged, but this assumes a constant difference of temperature between the ends of the bars. Now, in the case of the increased sectional area, the radiation of heat upon the warm junction will be unchanged, while for a given temperature difference the flow of heat to the cool junction, both on account of ordinary heat conduction and the PELTIER action of the current, will be increased, and, thus, the warm junction will not become so warm; thus, the actual sensibility will be less. The loss of heat by radiation from the bars can only be less in consequence of the warm junction being cooler, and, thus, the conclusion remains true, that there is no limit, except that imposed by the difficulty of working the materials, to the smallness of the sectional area that should be used. I may say here that I have found no great difficulty in making the bars as little as $\frac{1}{8}$ mm. thick, and in making perfect soldered joints where the weight of solder used does not exceed the fifth part of a milligram.

But to do this the ordinary methods must be discarded, and special means and special tools devised, when the difficulty becomes greatly reduced. Though full instructions would be interesting to the few who would ever probably care to make circuits of this extreme fineness, I do not think a detailed account of the manipulation would be suitable for insertion in this paper. I may, however, mention two causes, ordinarily of no great moment, which become, under the peculiar circumstances, of the first importance. One is the apparently instantaneous conduction of heat, and the other the surface tension of melted solder, which, unless provided against, will produce troublesome and unexpected results. There is no need to do more than mention the fusibility of the bismuth in the presence of the melted solder.

If the length of the bars be supposed increased in the ratio of $1 : n$, K will become nK, and C will become nC; thus, since $E \propto 1/\sqrt{CK}$, E will become n times as small. But here again less heat will reach the cool junction with the larger bars, both by conduction and by the Peltier action of the current. Thus, the warm junction will become warmer, and the cool junction colder: now, should the temperature difference become also n times as great, the actual sensibility of the circuit would remain unchanged. If no heat were radiated from the bars, then the temperature difference would be proportional to n, and the actual sensibility independent of n; but, on account of the radiation which must occur, the temperature difference would vary in a less ratio than $1 : n$, and therefore the bars could not be too short until the cold junction became sufficiently near the hot junction for it to be impossible to prevent the radiant heat from falling on it also.

On the other hand, on account of the increased flow of heat with the shorter bars, the cool junction would be made warmer, and the whole junction would therefore become warmer, and so there would be an increased loss by radiation from the warm junction. On this account the temperature difference would be less.

I may mention here that, in the narrow pattern of instrument, I have found it advantageous to make a special heat-receiving surface of the thinnest copper, of the size and shape suited to the purpose for which the instrument is made, and to keep the whole of the bars screened from the radiant heat altogether. This an adaptation to the radio-micrometer of the copper-faced thermopile, which Lord Rosse has found, and which is, obviously, so far preferable to the ordinary construction.

Though it is impossible to find by calculation the exact relative value of the three sources of equalisation of temperature in the circuit—namely, conduction of heat, Peltier effect, and radiation—it will be some guide to find, as far as data will allow, what their values are. It is most convenient to express them all by giving the time that would elapse before all the heat which is transferred to the cold junction by either of the first two actions, or which escapes in consequence of the third, would be sufficient to raise the pair of bars to the temperature of the warm junction, or, in the case of radiation, to raise it about half as much.

It is easy to show that this time is for the PELTIER effect equal to $JRs/t\theta$, where

J is JOULE'S equivalent ($4\cdot2 \times 10^7$),
$R = 2C$ is the resistance of the whole circuit ($67\cdot42 \times 10^6$),
s is the heat capacity of the two bars (·000798),
t is the mean absolute temperature (taken as 290),
θ the thermo-electric power (taken as 10,000).

The time of equalisation at a supposed constant rate is on this account 77·9 seconds.

This is true of the circuit of greatest efficacy; if the circuit of reduced size is employed, that is, one with a length one-N^{th} of this, then the total resistance R and the time of equalisation will be $(N + 1)/2N$ times as great. This can never be less than $\frac{1}{2}$.

The corresponding time for the equalisation by conduction may be taken as equal to $sl/D\alpha$, where

l is the length of the bars,
α their sectional area (separately),
D the sum of the conductivities of antimony and bismuth, *i.e.*, ·0607.

The value of this time is 2·63 seconds.

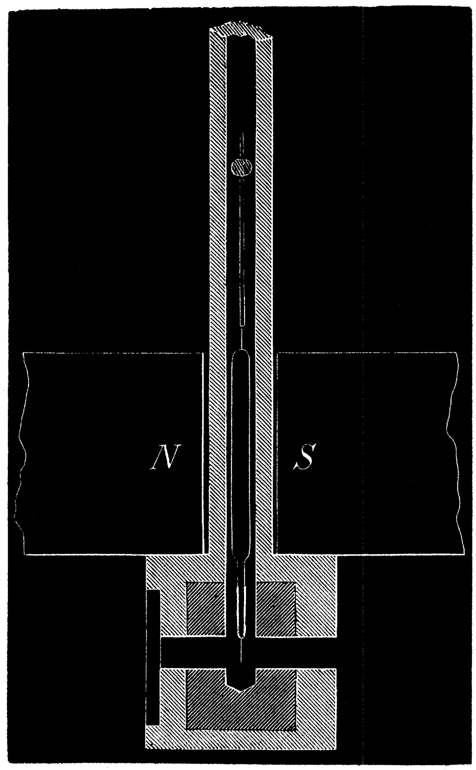
Thus, conduction appears to be far more important than the PELTIER effect, which, practically, may be left out of account. It can only become comparable when so much heat is lost by radiation that the rate of conduction at the cool end is far less than at the hot end, but in this case they would neither be of any practical importance in comparison with the radiation.

The data for finding the time for the escape of half the quantity of heat by radiation, contact of air, &c., are of doubtful value, on account of the very small size of the bars. But taking Professor TAIT'S figure, given in LUPTON'S Tables, for a black surface, the time would be about 29 seconds. Though no great value must be attached to this figure, it would appear that conduction is the main cause of the equalisation of the temperature of the circuit.

The PELTIER effect is involved in another manner in the action of the instrument. It must make a difference in the value of the least magnetic field which is necessary for the dead beat conditions. Thus, during motion of the circuit, currents are induced which oppose the motion; but these currents set up differences of temperature, which oppose the currents. Therefore, a stronger field may be employed before the dead beat conditions are reached. It is hardly necessary to do more than state that, as the result of calculation, no appreciable change is made in the value of the dead beat magnetic field on this account.

I must here mention the only difficulty which is apt to be found in practice. It arises from the magnetic properties of many materials which, insignificant though they are under ordinary methods of observation, become of serious importance when the extremely feeble force due to the torsion of the fibre is taken into account. I have found it absolutely necessary to sink the antimony and bismuth into a little well made by drilling a hole, no larger than necessary, in a piece of soft iron which is buried in the brasswork. A small lateral hole allows the radiation to fall on the heat-receiving surface. This effectually screens off the part which produces the greatest disturbance (see fig. 7, in which the iron is represented by the darker shading). The

Fig. 7.



copper must be exposed to the magnetism; therefore, it must be carefully examined to see that it is neutral, and it must then be kept away from emery or magnetic cleaning materials.

It is finally necessary to show that advantage is gained by employing the antimony-bismuth-copper combination, instead of the plain pair of wires used by M. D'ARSONVAL. While antimony-bismuth wires would, on account of their great thermo-electric power, be superior to palladium-silver, they would, on account of their magnetic qualities, disturb the natural period of the circuit. By the combination which I have employed, I am able to make use of this great thermo-electric power at the same time that the magnetic disturbance is avoided.

Some of the conclusions enunciated in the preliminary note require modification in view of the more extended investigation described in this paper. In a note, added March 23rd, I had concluded that more than one junction would be advantageous, but this is not the case.

But the point that requires special correction is the estimate formed of the greatest possible sensibility. This was calculated on the assumption that an instrument could be used practically when a particular circuit had a natural vibration period of 20 seconds, and was suspended in a field of 10,000 units, produced by an electro-magnet. Without entering on the question whether an electro-magnet could be used, it is sufficient to say that the resistance to the motion would be so great, the instrument would be so much more than just dead beat, that the time of coming to rest would be enormously prolonged. Thus, though the figure given is correct, the conditions to which it refers would not practically be advantageous.

It is, then, with some satisfaction that I turn to the result given by an instrument of the narrow form, having the best proportions. I have now taken quantities which can not only be separately obtained, but which can be used together, and which I have actually used with success.

Under these conditions, the least difference of temperature that could be observed with certainty, that is, one giving a movement of the light of $\frac{1}{5}$ mm. on the scale, would be due to a temperature difference of less than one two-millionth of a degree Centigrade. This figure is obtained by putting in the values of the quantities in the formula for temperature, which may be expressed in a variety of ways. The following are convenient :—

$$\text{Temperature difference} = \frac{128\pi^2\alpha'\sqrt{\text{KC}uv}}{3\text{T}^2\text{H}\theta},$$

or, if the dead beat magnetic field for such a circuit be employed as well,

$$\text{Temperature difference} = \tfrac{16}{3}\frac{\alpha'}{\theta}\frac{\pi}{\text{T}}\sqrt{\text{KC}}.$$

α' is the least observable angle of deflection (supposed $\frac{1}{10,000}$),

θ, the thermo-electric power (supposed 10,000),

T, the natural vibration period (supposed 10 seconds),

K, C, u, and v as before.

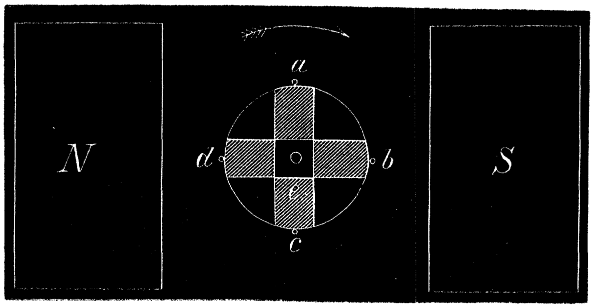
The temperature difference $8{\cdot}06 \times 10^{-7}$, found from the equation above, must be multiplied by $\frac{4}{7}$ to give the corresponding figures for the circuit of reduced length.

Tests made with an instrument of the narrow form, in which it was evident that the magnet still acted to a slight extent on the materials of the suspended portion, show it to be in practice exceedingly sensitive, and, what is of even more importance, the equilibrium of the moving parts remains perfectly stable. The surface which receives the radiant heat is in a particular case a disc only 2 mm. in diameter, and when the scale is 30 inches from the mirror, the hand held about a yard from the instrument produces at once a deflection of 16 cm. A candle flame at 9 feet produced a deviation of 45 mm. every time a small shutter close to the candle was pulled on one side with a piece of cotton, nothing else being allowed to move. Making a strict comparison between this result and those referred to in the preliminary note, this would give 1530 feet as the distance to which a halfpenny might be taken from a candle flame before the heat which it would receive would, if concentrated on the sensitive surface, be too small to produce a deflection of $\frac{1}{4}$ mm. This figure is considerably in excess of that obtained before, even though the fibre is 10 instead of 38 cm. long, the scale is only 30 inches from the screen, and a deflection of $\frac{1}{4}$ instead of $\frac{1}{10}$ mm. is here assumed as the least that could be observed with certainty.

With regard to the rotating pile described at the end of the preliminary note, I ought to say that something very similar is mentioned in NOAD's 'Electricity and Magnetism,' but I was not aware of this at the time of publication. There is, however, a curious difference, which is worth pointing out. So far as I have been able to learn, the wire frames described in NOAD's book only rotate one way when on one pole, and the other way when on the other pole of a magnet. Now, my arrangement will not rotate at all when placed over a pole; it will only rotate when between two poles, and then it will go either way when the heat is applied on one side, but will be prevented from moving when the heat is applied on the other side. Though the matter is of little importance, I may perhaps explain the reason for the peculiar behaviour of my arrangement.

The cross seen in fig. 8 is made with bismuth arms and an antimony centre. At the

Fig. 8.



ends of the arms four copper wires, *a*, *b*, *c*, *d*, are soldered at right angles to the plane of the cross, and lower down to a ring of copper. The whole is balanced on a point between the poles of a magnet, and is free to turn.

If heat is applied to the point *e*, an up current in the wire is produced at *c*, and a down current at *a*. Hence, the position of the cross is one of unstable equilibrium. Whichever way the cross begins to move, it will be kept moving in this direction. Suppose it to start in the direction of the arrow, that side of the antimony centre which faces the north pole will become the hottest, though it is gaining heat most rapidly when passing e; hence, the up current at d and the down current at b will each be urged on in the same direction. If, however, the cross started in the other direction, the side nearest the south pole would become the hottest, the current would pass the other way, and the reversed motion would still be kept up. Thus, whichever way it moves, it is kept moving in the same direction. If, however, the heat is applied on the opposite side, the first current produced is in a position of stable equilibrium; hence, the circuit shows no disposition to move unless there is want of symmetry, when it moves 45°, and then the equilibrium is absolutely stable. Whichever way the circuit is now made to turn, the direction of the currents will be such as stop the motion, for the same reason that in the previous case they maintained it.

In conclusion, the principal advantages in the instrument the development of which is described in this paper are:—

Extreme quickness and sensibility.

Freedom from extraneous thermal and magnetic influence.

The sensibility can be varied at will.

The instrument may be made dead beat, or its logarithmic decrement may be varied at will.

By the use of the quartz fibre, difficulties caused by the uncertain behaviour of silk under varying conditions of temperature and moisture—difficulties that would be far greater than in the case of a galvanometer—are completely obviated.

On the other hand, a disadvantage inherent in the instrument is that it must, like a galvanometer, be fixed in position; it is inferior to the thermopile or bolometer in the ease with which they can be pointed in any desired direction.